§ 3.10. The technique of tripodal transport

In the present §3.10, we *re-examine* **inter-universal Teichmüller theory** once again, this time from the point of view of the technique of **tripodal transport**. Various versions of this technique may also be seen in previous work of the author concerning

· ***p*-adic Teichmüller theory**,
· **scheme-theoretic Hodge-Arakelov theory**, and
· **combinatorial anabelian geometry**.

The proof given by

· **Bogomolov** [cf. [ABKP], [Zh], [BogIUT], as well as the discussion of §4.3, (iii), below] of the *geometric version of the Szpiro Conjecture* over the *complex numbers*

may also be re-interpreted from the point of view of this technique.

(i) **The notion of tripodal transport:** The general notion of **tripodal transport** may be summarized as follows [cf. also Fig. 3.21 below]:

(1^{trp}) One starts with a **"nontrivial property"** of interest [i.e., that one wishes to *verify*!] associated to some sort of *given* **arithmetic holomorphic structure** — such as a *hyperbolic curve* or a *number field* [cf. the discussion of §2.7, (vii)]. ring/scheme-th,

(2^{trp}) One observes that this **nontrivial property** of interest [i.e., associated to the given arithmetic holomorphic structure] may be derived by combining a **"relatively trivial" property**, again associated to the given arithmetic holomorphic structure, with some sort of **alternative property** of interest.

(3^{trp}) One establishes some sort of **parallel transport mechanism** — which is typically *not compatible* with the *given arithmetic [i.e., scheme-/ring-theoretic!] holomorphic structure* — that allows one to reduce the issue of verifying the **alternative property** of interest for the given arithmetic holomorphic structure to a "corresponding version" in the case of the **tripod** [i.e., the projective line minus three points] of this alternative property of interest.

(4^{trp}) One verifies the **alternative property** of interest in the case of the **tripod**.

(5^{trp}) By combining (1^{trp}), (2^{trp}), (3^{trp}), (4^{trp}), one concludes that the original **nontrivial property** of interest associated to the **given arithmetic holomorphic structure** does indeed hold, as desired.

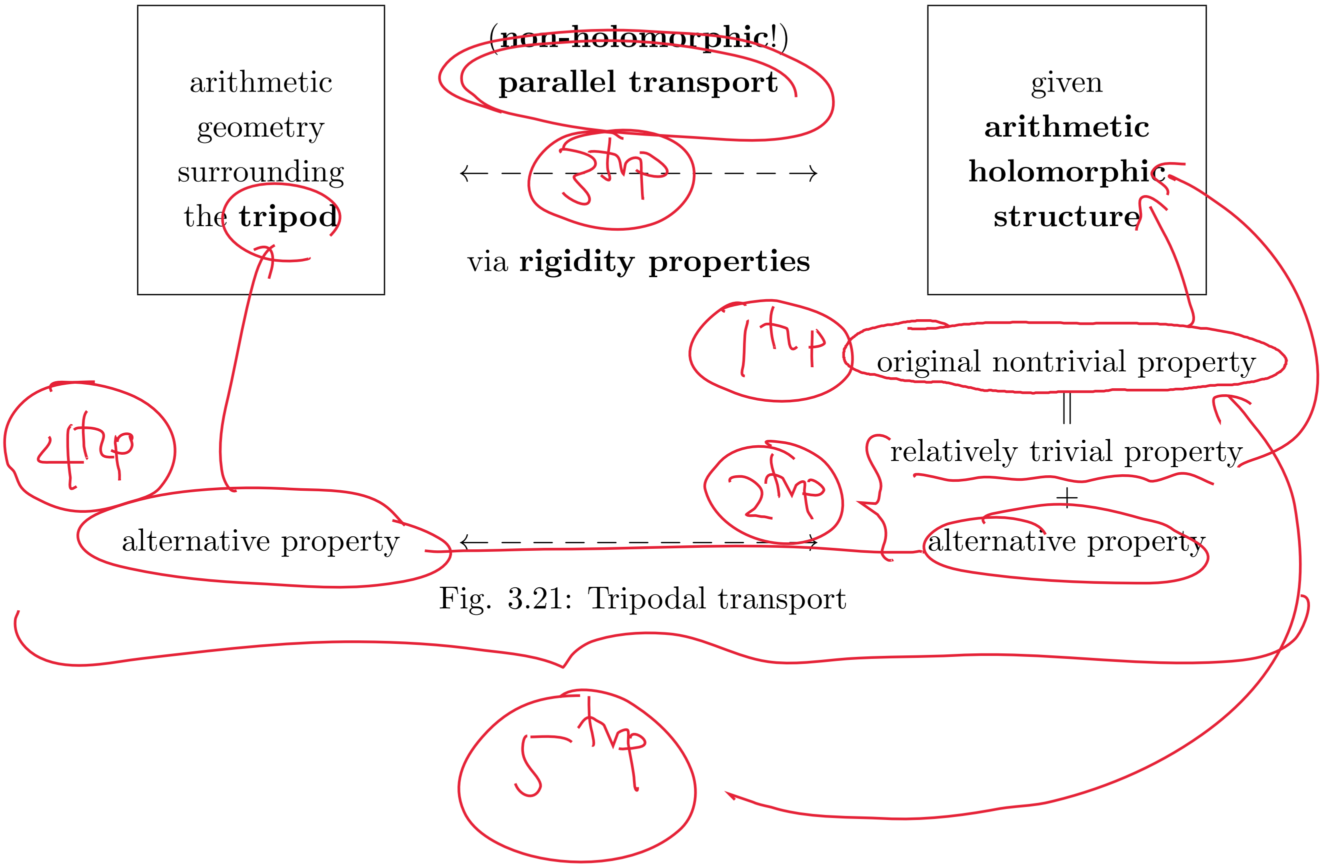


Fig. 3.21: Tripodal transport

Here, we note that the steps (3^{trp}), (4^{trp}) are often *very closely related*, and, indeed, at times, it is *difficult to isolate* these two steps from one another. This sort of argument might strike some readers at first glance as *"mysterious"* or *"astonishing"* in the sense that ultimately, one is able to

> conclude the original **nontrivial property** of interest [cf. (1^{trp})] associated to the **given arithmetic holomorphic structure** [cf. (5^{trp})] despite the fact that the **nontrivial content** of the argument centers around the **arithmetic** surrounding the **tripod** [cf. (3^{trp}), (4^{trp})], in sharp contrast to the fact that the argument only requires the use of a **"relatively trivial"** observation concerning the given arithmetic holomorphic structure [cf. (2^{trp})].

Perhaps it is most natural to regard this sense of *"mysteriousness"* or *"astonishment"* as a reflection of the *potency* of the **parallel transport mechanism** [cf. (3^{trp})] that is employed. This "potency" is, in many of the examples discussed below, derived as a consequence of various **rigidity properties**, such as **anabelian properties**. Such rigidity properties may only be derived by applying the mechanism of **parallel transport via rigidity properties** — *not* to relatively simple "types of mathematical objects" such as *vector spaces* or *modules*, as is typically the case in classical instances of parallel transport! — but rather to **complicated mathematical objects** [cf. the discussion of [IUTchIV], Remark 3.3.2], such as the sort of Galois groups/étale fundamental groups that occur in **anabelian geometry**, i.e., mathematical objects whose *intrinsic structure* is *sufficiently rich* to allow one to establish *rigidity properties* that are **sufficiently "potent"** *to compensate for the "loss of structure"* that arises from *sacrificing compatibility with classical scheme-/ring-theoretic structures.*

Here, we note that it is necessary to sacrifice compatibility with classical scheme-/ring-theoretic structures precisely because such structures typically constitute a **fundamental obstruction** to relating the arithmetic surrounding the *given arithmetic holomorphic structure* to the arithmetic surrounding the *tripod.* A typical example of this sort of "fundamental obstruction" may be seen by considering, for instance, the case of *two [scheme-theoretically!] non-isomorphic proper hyperbolic curves* over an algebraically closed field of characteristic zero, which, nonetheless, have *isomorphic étale fundamental groups.* This point of view, i.e., of overcoming the sort of *"fundamental obstruction" to parallel transport* that arises from *imposing the restriction of working within a fixed scheme/ring theory*, is closely related to the introduction of the notions of **Frobenius-like** and **étale-like** structures — cf. the discussion of §2.7, (ii), (iii); §2.8.

<u>Additional observations</u>:

· The technique of tripodal transport constitutes a **unifying theme**/**efficient mechanism** for documenting the **remarkably similar conceptual framework** underlying various aspects of my work since the early 1990's.

· The technique of tripodal transport shows explicitly how **anabelian geometry** is by no means an isolated topic, but rather a topic that **inextricably intertwined** with the rest of arithmetic geometry.

· The **parallel transport** portion of tripodal transport may be understood as a *mechanism* for documenting structures **common** — i.e., "∧" — to **distinct holomorphic structures**. Such common structures may then be used to construct **"containers"** for the distinct holomorphic structures that allow one to **compare** the distinct holomorphic structures. One prime example of such a *mechanism* is **anabelian geometry**!

· The technique of tripodal transport may be regarded as a reflection of the the fact that the **tripod** may be regarded as a sort of **geometric representation** of the **intertwining** between **addition** and **multiplication** in a ring structure — cf. the theory of BGT! — hence is of *fundamental importance* to arithmetic geometry.

(ii) **Inter-universal Teichmüller theory via tripodal transport:** We begin our discussion by observing that, when viewed from the point of view of the notion of tripodal transport, **inter-universal Teichmüller theory** may be recapitulated as follows:

→ in RHS of Θ-link

> The fundamental **log volume estimate** (12^{est}) [cf. (1^{trp})] is obtained in the argument discussed in §3.7, (ii) [cf. (5^{trp})], by combining [cf. (9^{est}), (10^{est}), (11^{est})] a *relatively simple* argument [cf. (2^{trp})] carried out in the **arithmetic holomorphic structure** of the **RHS** of the Θ-link [cf. (1^{est}), (7^{est})], involving *relatively simple* operations such as the formation of the **holomorphic hull** [cf. (6^{est}), (7^{est}), (8^{est})], with the **parallel transport mechanism** [cf. (3^{trp}), as well as the discussion of §3.1, (iv), (v)] furnished by the **multiradial representation** [cf. (2^{est}), (3^{est}), (4^{est}), (5^{est})], which is established by considering various properties of objects [cf. §3.4, §3.6], such as the **theta function** on the **Tate curve** [cf. §3.4, (iii), (iv); Fig. 3.9], on the **LHS** of the Θ-link [cf. (4^{trp})].

Here, we recall from the discussion of §3.4, (iii), (iv); Fig. 3.9, that

> the theory surrounding the **theta function** on the **Tate curve** may be thought of as a sort of **function-theoretic representation** of the *p-adic arithmetic geometry* of a copy of the **tripod** for which the cusps "0" and "∞" are subject to the involution symmetry that permutes these two cusps and leaves the cusp "1" fixed.

Also, we recall from the discussion of §3.7, (i) [cf. also the discussion of the properties "IPL", "SHE", "APT", "HIS" in [IUTchIII], Remark 3.11.1] that the *parallel transport mechanism* furnished by the *multiradial representation* revolves around the following *central property*:

> the algorithm that yields the multiradial representation converts **any** collection of **input data** [i.e., not just the *codomain* data (a^q), (b^q), (c^q) of the Θ-link!] that is isomorphic to the *domain* data (a^Θ), (b^Θ), (c^Θ) of the Θ-link — i.e., in somewhat more technical terminology [cf. [IUTchII], Definition 4.9, (viii)], any $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip** — into **output data** that is *expressed* in terms of the **arithmetic holomorphic structure** of the **input data**, i.e., of the codomain of the Θ-link.

Finally, at a more technical level, we recall from §3.3, (vi); §3.4, (ii); §3.4, (iii), (iv), that this *parallel transport mechanism* is established by applying

- the theory of the **étale theta function** developed in [EtTh];
- the theory of [local and global] **mono-anabelian reconstruction** developed in [AbsTopII], [AbsTopIII].

Here, it is of interest to observe that both the theory of **elliptic cuspidalization**, which plays an important role in [EtTh], and the theory of **Belyi cuspidalization**, which plays an important role in [AbsTopII], [AbsTopIII], may be regarded as *essentially formal consequences* of the **fundamental anabelian results** obtained in [*p*GC]. The *rigidity properties* developed in [EtTh] also depend, in a fundamental way, on the *interpretation* [i.e., as rigidity properties of the desired type!] given in [EtTh] of the **theta symmetries** of the theta function on the Tate curve.

(iii) ***p*-adic Teichmüller theory via tripodal transport:** When viewed from the point of view of the notion of tripodal transport, a substantial portion of the ***p*-adic Teichmüller theory** of [pOrd], [pTch], [pTchIn] may be summarized as follows:

> One constructs a theory of **canonical indigenous bundles**, **canonical Frobenius liftings**, and associated **canonical Galois representations** into $PGL_2(-)$ [for a suitable "$(-)$" — cf. [pTchIn], Theorems 1.2, 1.4, for more details] for *quite general p-adic hyperbolic curves* [cf. (1^{trp}), (2^{trp}), (5^{trp})] by establishing a **parallel transport mechanism** [cf. (3^{trp})] that allows one to transport similar canonical objects associated to the *tautological family of elliptic curves* over the **tripod** [cf. (4^{trp})].

Here, we recall that, prior to [pOrd], the existence of such canonical objects associated to a p-adic hyperbolic curve was only known in the case of **Shimura curves**, i.e., such as the **tripod**. From the point of view of the notion of *tripodal transport*, it is also of interest to observe that:

> The notion of an **ordinary Frobenius lifting** [cf. [pTchIn], Theorem 1.3], which plays a *central role* in [pOrd], [pTch], [pTchIn], may be understood as a sort of p-adic generalization of the most fundamental example of a **Frobenius lifting**, namely, the endomorphism
>
> $$T \mapsto T^p$$
>
> [where T denotes the standard coordinate on the projective line] of the **tripod** over a p-adic field. This endomorphism is *equivariant* with respect to the *symmetry* of the tripod which permutes the cusps "0" and "∞" and leaves the cusp "1" fixed.

At a more technical level, we recall that the *parallel transport mechanism* employed in p-adic Teichmüller theory revolves around the following *two fundamental technical tools*:

- · the fact that the natural morphism from the **moduli stack of nilcurves** [i.e., pointed stable curves equipped with an indigenous bundle whose p-curvature is square nilpotent] to the corresponding moduli stack of pointed stable curves is a **finite**, **flat**, and **local complete intersection morphism** of degree p to the power of the dimension of these moduli stacks [cf. [pTchIn], Theorem 1.1];
- · various **strong rigidity properties**, with respect to **deformation**, that hold precisely over the **ordinary locus** of the *moduli stack of nilcurves*, i.e., the *étale locus* of the natural morphism from the moduli stack of nilcurves to the corresponding moduli stack of pointed stable curves.

In this context, it is of interest to observe, considering the fundamental role played by such notions as *differentials* and *curvature* in the *classical differential-geometric version of parallel transport*, that both of these fundamental technical tools rely on various subtle properties of the ***p*-curvature** and **Frobenius actions on differentials**. This relationship with *differentials* is also interesting from the point of view of the fundamental role played by the theory of [pGC] in the discussion of [EtTh], [AbsTopII], and [AbsTopIII] in (ii), since **differentials**, treated from a ***p*-adic Hodge-theoretic** point of view, play a fundamental role in [pGC]. Finally, we observe that although anabelian results do not play *any role* in the parallel transport mechanism of p-adic Teichmüller theory, it is interesting to note that p-adic Teichmüller theory has an *important application* to **absolute** mono-**anabelian geometry** [cf. [CanLift], §3, as well as the discussion of [IUTchI], §I4; [IUTchII], Remark 4.11.4, (iii)].

(iv) **Scheme-theoretic Hodge-Arakelov theory via tripodal transport:** When viewed from the point of view of the notion of tripodal transport, the **fundamental theorem of Hodge-Arakelov theory**, i.e., the **natural isomorphism** reviewed at the beginning of §3.9, (i) [cf. also Example 2.14.3; [HASurI]; [HASurII]], may be understood as follows:

> One verifies [cf. the discussion of [HASurI], §1.1] that the *natural morphism* obtained by *evaluating sections of an ample line bundle over the universal vectorial extension of an elliptic curve at torsion points* [cf. the discussion at the beginning of Example 2.14.3] is indeed an **isomorphism** [cf. (1^{trp}), (5^{trp})] by verifying that it is an isomorphism in the case of **Tate curves** by means of an explicit computation involving **derivatives** of **theta functions** [cf. (4^{trp})] and then proceeding to **parallel transport** this isomorphism in the case of Tate curves to the entire compactified moduli stack of elliptic curves in characteristic 0 by means of an explicit computation [the *leading term* portion of which is reviewed in §3.9, (i)] of the **degrees** of the vector bundles on this compactified moduli stack that constitute the domain and codomain of the natural morphism under consideration [cf. (2^{trp}), (3^{trp})].

Here, we recall from the discussion of (ii) above; §3.4, (iii), (iv); Fig. 3.9, that

> the theory surrounding the **theta function** on the **Tate curve** may be thought of as a sort of **function-theoretic representation** of the *[not necessarily p-!]adic arithmetic geometry* of a copy of the **tripod** for which the cusps "0" and "∞" are subject to the involution symmetry that permutes these two cusps and leaves the cusp "1" fixed.

In this context, it is also perhaps of interest to recall that there is an alternative approach to the *parallel transport mechanism* discussed above [i.e., computing *degrees* of vector bundles on the compactified moduli stack of elliptic curves], namely, the *parallel transport mechanism* applied ~~in the proof of [HASurII], Theorem~~ 4.3, which exploits various special properties of the **Frobenius** and **Verschiebung** morphisms in **positive characteristic**. Finally, ~~we observe that although the scheme~~-theoretic Hodge-Arakelov theory of [HASurI], [HASurII] is *not directly related, in a logical sense, to anabelian geometry*, it nevertheless ~~played a central role, as was~~ discussed in detail in §3.9, in motivating the development of **inter-universal Teichmüller theory**, which may be understood as a sort of *reformulation of ~~the essential content of the scheme~~-theoretic Hodge-Arakelov theory of [HASurI], [HASurII] via techniques based on* **anabelian geometry**.

via degrees

ω_E-torsor

$\in H^1(E, \omega_E)$

Example 2.14.3. Finite discrete approximation of harmonic analysis on complex tori. Let E be an elliptic curve over a field F of characteristic zero, $E^\dagger \to E$ the *universal extension* of E, $\eta \in E(F)$ a [nontrivial] torsion point of order 2, $l \neq 2$ a prime number. Write $E[l] \subseteq E$ for the subscheme of *l-torsion points*, $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(l \cdot [\eta])$ [where "$[\eta]$" denotes the effective divisor on E determined by η]. Here, we recall that $E^\dagger \to E$ is an $\mathbb{A}^1$*-torsor* [so $E[l]$ may also be regarded as the subscheme $\subseteq E^\dagger$ of l-torsion points of $E^\dagger$]. In particular, it makes sense to speak of the sections $\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{<l} \subseteq \Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})$ of $\mathcal{L}$ over $E^\dagger$ whose *relative degree*, with respect to the morphism $E^\dagger \to E$, is $< l$. Then the simplest version of the **fundamental theorem of Hodge-Arakelov theory** states that **evaluation** at the subscheme of ***l*-torsion points** $E[l] \subseteq E^\dagger$ yields a *natural isomorphism of F-vector spaces of dimension* l^2

$$\boxed{\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{<l} \xrightarrow{\sim} \mathcal{L}|_{E[l]}}$$

[cf. [HASurI], Theorem A$^{\text{simple}}$].

$\dim_F = l \cdot \dim_F \underbrace{\Gamma(E, \mathcal{L})}_{F\text{-}\dim\ l}$

$= l^2$

$\mathrm{Out}(\mathrm{gp.}) := \mathrm{Aut}(\mathrm{gp})/\mathrm{Inn}$

(v) **Combinatorial anabelian geometry via tripodal transport:** Let F be a *number field*, $\overline{F}$ an algebraic closure of F, X a *hyperbolic curve* over F, $n \geq 1$ an integer. Write X_n for the *n-th configuration space* of X [cf., e.g., [MT], Definition 2.1, (i)]; Π_n for the *étale fundamental group* of $X_n \times_F \overline{F}$ [for a suitable choice of basepoint]; $\Pi \stackrel{\mathrm{def}}{=} \Pi_1$; Π^{tpd} for "Π" in the case where X is the *tripod* [i.e., the projective line minus three points]; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F)$; $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ for the group of outer automorphisms of Π_n satisfying certain technical conditions [i.e., "FC"] involving the *fiberwise subgroups* and *cuspidal inertia subgroups* [cf. [CombCusp], Definition 1.1, (ii), for more details]. Thus, it follows from the definition of "$\mathrm{Out}^{\mathrm{FC}}$" that the natural projection $X_{n+1} \to X_n$ given by forgetting the $(n+1)$-st factor determines a homomorphism

$$\phi_{n+1} : \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

[cf. the situation discussed in [NodNon], Theorem B]; the natural action of G_F on $X_n \times_F \overline{F}$ determines an *outer Galois representation*

$$\boxed{\rho_n : G_F \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)}$$

[cf. the situation discussed in [NodNon], Theorem C]. Write $\rho \stackrel{\mathrm{def}}{=} \rho_1$, ρ^{tpd} for "ρ" in the case where X is the *tripod*.

$1 \to \Pi_n \to \pi_1(X_n) \to G_F \to 1$

Then

> the proof of the **injectivity** [cf. [NodNon], Theorem C] of
>
> $$\boxed{\rho : G_F \ \to \ \mathrm{Out}^{\mathrm{FC}}(\Pi)} \ \to \text{assoc'd } X/F$$
>
> given in [NodNon] is perhaps the most **transparent/prototypical** example of the phenomenon of **tripodal transport**.

Indeed, this proof may be summarized as follows:

> One makes the ["relatively trivial"! — cf. (2^{trp})] *observation* that ρ admits a *factorization*
>
> $$\boxed{\rho = \phi_2 \circ \phi_3 \circ \rho_3 : G_F \xrightarrow{\rho_3} \mathrm{Out}^{\mathrm{FC}}(\Pi_3) \underbrace{\xrightarrow{\varphi_3} \mathrm{Out}^{\mathrm{FC}}(\Pi_2) \xrightarrow{\varphi_2}}_{\varphi_{23}} \mathrm{Out}^{\mathrm{FC}}(\Pi)}$$
>
> — which allows one to *reduce* [cf. (2^{trp})] the verification of the *desired injectivity* of ρ [cf. (1^{trp}), (5^{trp})] to the verification of the *injectivity* of $\phi_{23} \stackrel{\mathrm{def}}{=} \phi_2 \circ \phi_3$ [cf. (3^{trp}), (4^{trp})] and ρ_3 [cf. (4^{trp})].

Then:

- One observes that the *injectivity* of ϕ_{23} depends only on the type "(g, r)" [i.e., the genus and number of punctures] of X, hence may be verified in the case of — i.e., may be **"parallel transported"** [cf. (3^{trp})] to the case of — a **totally degenerate pointed stable curve**, i.e., a pointed curve obtained by gluing together some collection of **tripods** along the various cusps of the tripods [cf. (4^{trp})]. On the other hand, in the case of such a totally degenerate pointed stable curve, the *desired injectivity* [i.e., of the analogue of "ϕ_{23}"] may be verified by applying the purely combinatorial/group-theoretic techniques of **combinatorial anabelian geometry** developed in [CombCusp], [NodNon] [cf. [NodNon], Theorem B].
- One verifies the *injectivity* of ρ_3 by applying a certain natural homomorphism called the **tripod homomorphism**

$$\tau : \mathrm{Out}^{\mathrm{FC}}(\Pi_3) \to \mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{tpd}})$$

[cf. [CbTpII], Theorem C, (ii)], which satisfies the property that $\rho^{\mathrm{tpd}} = \tau \circ \rho_3 : G_F \to \mathrm{Out}^{\mathrm{FC}}(\Pi_3) \to \mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{tpd}})$ and hence allows one to conclude the injectivity of ρ_3 from the well-known **injectivity result of Belyi** to the effect that ρ^{tpd} is injective [cf. (4^{trp})].

Here, it is interesting to note, especially in light of the discussion of *anabelian results* and *differentials* in the final portions of (i), (ii), (iii), the *central role* played by **combinatorial anabelian geometry** — i.e., in particular, various *combinatorial versions of the Grothendieck Conjecture* such as [NodNon], Theorem A — in the **parallel transport mechanism** discussed above. Such combinatorial versions of the Grothendieck Conjecture concern *group-theoretic characterizations* of the *decomposition* of a pointed stable curve into various *irreducible components* glued together along the *nodes* of the curve. This sort of decomposition may be interpreted as a sort of *discrete version* of the notion of a *differential*, i.e., which may be thought of as a *decomposition of a ring/scheme structure into infinitesimals*. Finally, we emphasize that this proof of the *injectivity of* ρ is a **particularly striking** example of the phenomenon of **tripodal transport**, in the sense that the issue of relating the injectivity of ρ for an *arbitrary* X to the injectivity of ρ^{tpd}, i.e., in the case of the *tripod*, seems, a priori, to be *entirely intractable*, at least so long as one restricts oneself to morphisms between *schemes* [cf. the discussion in the final portion of (i)].

of Szpiro Conj. /ℂ

(vi) **Tripodal transport and Bogomolov's proof:** Often, as in the examples discussed in (ii), (iii), (iv), above, the **tripod** that appears in instances of the phenomenon of tripodal transport is a tripod in which the *cusps "0" and "∞"* play a *distinguished*, but *symmetric role*, which is somewhat different from the role played by the *cusp "1"*. When considered from this point of view, the tripod may thought of as the underlying scheme of the group scheme $\mathbb{G}_{\mathrm{m}}$ [with its origin removed], hence, in particular, as a sort of algebraic version of the **topological circle** $\mathbb{S}^1$. If one thinks of the tripod in this way, i.e., as corresponding to $\mathbb{S}^1$, then the proof given by

> **Bogomolov** [cf. [ABKP], [Zh], [BogIUT], as well as the discussion of §4.3, (iii), below] of the *geometric version of the Szpiro Conjecture* over the *complex numbers* may also be understood as an instance, albeit in a somewhat generalized sense, of the technique of **tripodal transport**.

arith. hol. strs.

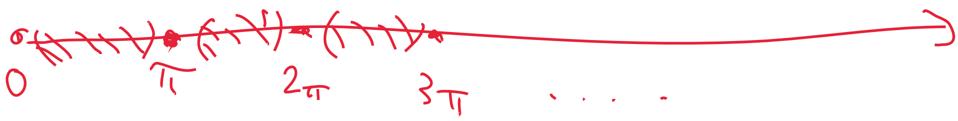
$\longleftrightarrow$ hol. strs.

To explain further, we introduce notation as follows:

- Write $\mathrm{Aut}_\pi(\mathbb{R})$ for the group of *self-homeomorphisms* $\mathbb{R} \xrightarrow{\sim} \mathbb{R}$ that *commute* with translation by $\pi \in \mathbb{R}$. Thus, if we think of $\mathbb{S}^1$ as the *quotient* $\mathbb{R}/(2\pi \cdot \mathbb{Z})$, then $\mathrm{Aut}_\pi(\mathbb{R})$ may be understood as the group of self-homeomorphisms of $\mathbb{R}$ that *lift* elements of the group $\mathrm{Aut}_+(\mathbb{S}^1)$ of orientation-preserving self-homeomorphisms of $\mathbb{S}^1$ that commute with multiplication by -1 on $\mathbb{S}^1$. In particular, we have a *natural exact sequence* $1 \to 2\pi \cdot \mathbb{Z} \to \mathrm{Aut}_\pi(\mathbb{R}) \to \mathrm{Aut}_+(\mathbb{S}^1) \to 1$.
- Write $\mathrm{Aut}_\pi(\mathbb{R}_{\geq 0})$ for the group of *self-homeomorphisms* $\mathbb{R}_{\geq 0} \xrightarrow{\sim} \mathbb{R}_{\geq 0}$ that *stabilize* and *restrict to the identity* on the subset $\pi \cdot \mathbb{N} \subseteq \mathbb{R}_{\geq 0}$.
- Write $\mathbb{R}_{|\pi|}$ for the set of $\mathrm{Aut}_\pi(\mathbb{R}_{\geq 0})$-*orbits* of $\mathbb{R}_{\geq 0}$ [relative to the natural action of $\mathrm{Aut}_\pi(\mathbb{R}_{\geq 0})$ on $\mathbb{R}_{\geq 0}$]. Thus,

$$\mathbb{R}_{|\pi|} = \left(\bigcup_{n \in \mathbb{N}} \ \{ \ [n \cdot \pi] \ \} \right) \ \cup \ \left(\bigcup_{m \in \mathbb{N}} \ \{ \ [(m \cdot \pi, (m+1) \cdot \pi)] \ \} \ \right)$$

— where we use the notation "[−]" to denote the element in $\mathbb{R}_{|\pi|}$ determined by an element or nonempty subset of $\mathbb{R}_{\geq 0}$ that lies in a single $\mathrm{Aut}_\pi(\mathbb{R}_{\geq 0})$-orbit; we use the notation "(−, −)" to denote an open interval in $\mathbb{R}_{\geq 0}$; we observe that the natural order relation on $\mathbb{R}_{\geq 0}$ induces a *natural order relation* on $\mathbb{R}_{|\pi|}$.



· Write $\delta^{\sup} : \mathrm{Aut}_\pi(\mathbb{R}) \to \mathbb{R}_{|\pi|}$ for the map that assigns to $\alpha \in \mathrm{Aut}_\pi(\mathbb{R})$ the element $\sup(\delta(\alpha)) \in \mathbb{R}_{|\pi|}$, where we observe that

$|\beta^{-1}\alpha\beta(x) - \beta^{-1}\beta(x)|$

$$\delta(\alpha) \stackrel{\mathrm{def}}{=} \{ [\ |\alpha(x) - x|\ ] \mid x \in \mathbb{R} \} \subseteq \mathbb{R}_{|\pi|}$$

$\Uparrow$

is a *finite subset* [cf. the definition of $\mathrm{Aut}_\pi(\mathbb{R})$!] of $\mathbb{R}_{|\pi|}$, and that [as is easily verified, by observing that for any $\beta \in \mathrm{Aut}_\pi(\mathbb{R})$ and $x, y \in \mathbb{R}$ such that $x \leq y$, there exists a $\gamma \in \mathrm{Aut}_\pi(\mathbb{R}_{\geq 0})$ such that $\beta(y) - \beta(x) = \beta((y - x) + x) - \beta(x) = \gamma(y - x)$]

$|\beta^{-1}\alpha x - \beta^{-1}x|$

the assignments $\delta(-)$, $\delta^{\sup}(-)$ are $\mathrm{Aut}_\pi(\mathbb{R})$**-conjugacy invariant**.

· Write $SL_2(\mathbb{R})^{\sim}$ for *universal covering* of $SL_2(\mathbb{R})$. Thus, we have a *natural central extension of topological groups* $1 \to \mathbb{Z} \to SL_2(\mathbb{R})^{\sim} \to SL_2(\mathbb{R}) \to 1$. By composing the natural embedding $\mathbb{S}^1 \hookrightarrow \mathbb{R}^{2\times} \stackrel{\mathrm{def}}{=} \mathbb{R}^2 \setminus \{(0,0)\}$ with the natural projection $\mathbb{R}^{2\times} \twoheadrightarrow \mathbb{R}^{2\angle} \stackrel{\mathrm{def}}{=} \mathbb{R}^{2\times}/\mathbb{R}_{>0}$, we obtain a *natural homeomorphism* $\mathbb{S}^1 \stackrel{\sim}{\to} \mathbb{R}^{2\angle}$, hence [by considering the natural action of $SL_2(\mathbb{R})$ on $\mathbb{R}^{2\times}$, $\mathbb{R}^{2\angle}$] *natural actions*

$= |\alpha x - x|$

$$SL_2(\mathbb{R}) \curvearrowright \mathbb{S}^1; \quad SL_2(\mathbb{R})^{\sim} \curvearrowright \mathbb{R}$$

[where we think of $\mathbb{R}$ as the *universal covering* of $\mathbb{S}^1 = \mathbb{R}/(2\pi \cdot \mathbb{Z})$], the *latter* of which determines a *natural injective homomorphism*

$$SL_2(\mathbb{R})^{\sim} \hookrightarrow \mathrm{Aut}_\pi(\mathbb{R})$$

[which, at times, we shall use to think of $SL_2(\mathbb{R})^{\sim}$ as a *subgroup* of $\mathrm{Aut}_\pi(\mathbb{R})$]. We may assume without loss of generality that the generator "1" of $\mathbb{Z} \hookrightarrow SL_2(\mathbb{R})^{\sim}$ was chosen so as to act on $\mathbb{R}$ in the *positive direction*.

· Write $SL_2(\mathbb{Z})^{\sim} \stackrel{\text{def}}{=} SL_2(\mathbb{R})^{\sim} \times_{SL_2(\mathbb{R})} SL_2(\mathbb{Z})$. Thus, we have a *natural central extension of discrete groups* $1 \to \mathbb{Z} \to SL_2(\mathbb{Z})^{\sim} \to SL_2(\mathbb{Z}) \to 1$. One shows easily [e.g., by considering the *discriminant modular form*, as in [BogIUT]] that the *abelianization* of $SL_2(\mathbb{Z})^{\sim}$ is isomorphic to $\mathbb{Z}$, and hence that there exists a *unique surjective homomorphism*

$(SL_2(\mathbb{Z}))^{ab}$

$$\boxed{\chi : SL_2(\mathbb{Z})^{\sim} \twoheadrightarrow \mathbb{Z}}$$

$\omega^{\times} \to (\omega^{\times})^{\mathbb{R}} \simeq M_{ell} \times \mathbb{C}^{\times}$

$SL_2(\mathbb{Z})$ $\pi_1((M_{ell})_{\mathbb{C}})$ $\pi_1(\omega^{\times})$ $SL_2(\mathbb{Z})^{\sim}$

that maps positive elements of $\mathbb{Z} \hookrightarrow SL_2(\mathbb{Z})^{\sim}$ to positive elements of $\mathbb{Z}$.

12.

In some sense, the *fundamental phenomenon* that underlies Bogomolov's proof is the following *elementary fact*:

> Whereas the $SL_2(\mathbb{Z})$**-conjugacy classes** of the *unipotent elements*
>
> $$\tau^m \stackrel{\text{def}}{=} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$$
>
> **differ** for different positive integers m, the $SL_2(\mathbb{R})$**-conjugacy classes** of these elements **coincide** for different positive integers m.

$\hookrightarrow$ parallel transport (cf. (3rd))

In the context of Bogomolov's proof, if one thinks of $SL_2(\mathbb{Z})$ as the topological fundamental group of the *moduli stack of elliptic curves* over the complex numbers, then such unipotent elements arise as the images in $SL_2(\mathbb{Z})$ — via the [outer] homomorphism induced on topological fundamental groups by the classifying morphism associated to a *family of one-dimensional complex tori over a hyperbolic Riemann surface S of finite type* — of the natural generators of cuspidal inertia groups of the topological fundamental group of S. In this situation, the positive integer m then corresponds to the **valuation** of the ***q*-parameter** at a cusp of S. Next, we recall [cf., e.g., [BogIUT], (B1)] that **unipotent elements** of $SL_2(\mathbb{R})$ admit **canonical liftings** to $SL_2(\mathbb{R})^{\sim}$. In particular, it makes sense to apply both δ^{sup} and χ to the canonical lifting $\widetilde{\tau}^m \in SL_2(\mathbb{Z})^{\sim}$ of τ^m. Since χ is a **homomorphism**, we have

cpx, fin. no. of pts.

$SL(\mathbb{Z})^{\sim} \twoheadrightarrow \chi$

$$\chi(\widetilde{\tau}^m) = m$$

[cf., e.g., [BogIUT], (B3)]. On the other hand, since $\delta^{\text{sup}}(-)$ is $\text{Aut}_{\pi}(\mathbb{R})$**-** [hence, in particular, $SL_2(\mathbb{R})^{\sim}$**-**] **conjugacy invariant**, we have

$$\boxed{\delta^{\text{sup}}(\widetilde{\tau}^m) \ < \ [\pi]}$$

[cf., e.g., [BogIUT], (B1)] for *arbitrary m*.

It is precisely by applying both χ and $\delta^{\sup}(-)$ to a certain *natural relation* [arising from the image in $SL_2(\mathbb{Z})$ of the "usual defining relation" of the topological fundamental group of S] between elements $\in SL_2(\mathbb{Z})$ lifted to $SL_2(\mathbb{Z})^{\sim}$ that one is able to derive the **geometric version** of the **Szpiro inequality**, that is to say, to **bound** the **height** of the given family of one-dimensional complex tori — i.e., more concretely, in essence, the *sum of the "m"'s* arising from the various cusps of S [cf., e.g., [BogIUT], (B4)] — by a number that depends only on the genus and number of cusps of S and *not* on the "m"'s themselves [cf., e.g., [BogIUT], (B2), (B5)]. From the point of view of the technique of **tripodal transport**, one may summarize this argument as follows:

> one bounds the **height** [i.e., essentially, the sum of the "m"'s] of the given family of one-dimensional complex tori [cf. (5^{trp})] — which is a reflection of the **holomorphic moduli** of this family [cf. (1^{trp})] — by combining a "relatively trivial" [cf. (2^{trp})] object χ arising from the **holomorphic structure** of the moduli stack of elliptic curves over the complex numbers [i.e., from the *discriminant modular form*] with the **parallel transport mechanism** [cf. (3^{trp})] given by passing from the *"holomorphic"* $SL_2(\mathbb{Z})$, $SL_2(\mathbb{Z})^{\sim}$ to the *"real analytic"* $SL_2(\mathbb{R})$, $SL_2(\mathbb{R})^{\sim}$, i.e., in essence, by passing to the $\text{Aut}_+(\mathbb{S}^1)$**-invariant geometry of** $\mathbb{S}^1$, as reflected in the $\text{Aut}_\pi(\mathbb{R})$**-conjugacy invariant** map $\delta^{\sup} : \text{Aut}_\pi(\mathbb{R}) \to \mathbb{R}_{|\pi|}$ [cf. (4^{trp})].

From the point of view of the *analogy* [cf. the discussion of (ii) above; [BogIUT]] between *Bogomolov's proof* and *inter-universal Teichmüller theory*, we observe that:

- The **canonical lifts** discussed above of **unipotent elements** $\in SL_2(\mathbb{Z})$ to $SL_2(\mathbb{Z})^{\sim}$ correspond to the theory of the **étale theta function** [i.e., [EtTh]] in inter-universal Teichmüller theory.
- The $\mathrm{Aut}_+(\mathbb{S}^1)$**-invariant geometry of** $\mathbb{S}^1$, as reflected in the $\mathrm{Aut}_\pi(\mathbb{R})$**-conjugacy invariant** map $\delta^{\mathrm{sup}} : \mathrm{Aut}_\pi(\mathbb{R}) \to \mathbb{R}_{|\pi|}$, corresponds to the theory of **mono-analytic log-shells** and related **log-volume estimates** [cf. (12^{est}); §3.7, (iv); [IUTchIV], §1, §2] in inter-universal Teichmüller theory. In particular, $\mathrm{Aut}_+(\mathbb{S}^1)$**-/**$\mathrm{Aut}_\pi(\mathbb{R})$**-indeterminacies** in Bogomolov's proof — in which both the **additive** [i.e., corresponding to *unipotent subgroups* of $SL_2(\mathbb{R})$] and **multiplicative** [i.e., corresponding to *toral*, or equivalently, *compact subgroups* of $SL_2(\mathbb{R})$] dimensions of $SL_2(\mathbb{R})$ are **"confused"** within the **single dimension** of $\mathbb{S}^1$ — correspond to the **indeterminacies** (Ind1), (Ind2), (Ind3) of inter-universal Teichmüller theory.
- The role played by $SL_2(\mathbb{Z})$, $SL_2(\mathbb{Z})^{\sim}$, χ corresponds to the role played by the **fixed arithmetic holomorphic structure** of the **RHS** of the Θ-link [cf. (1^{est}), (6^{est}), (7^{est}), (8^{est}), (9^{est}), (10^{est}), (11^{est})] in the argument of §3.7, (ii). By contrast, the role played by $SL_2(\mathbb{R})$, $SL_2(\mathbb{R})^{\sim}$, $\delta^{\mathrm{sup}}(-)$ corresponds to the role played by the **multiradial representation** [cf. (2^{est}), (3^{est}), (4^{est}), (5^{est})] in the argument of §3.7, (ii).

In particular, one has natural *correspondences*

$$\begin{array}{rcl} SL_2(\mathbb{R}), SL_2(\mathbb{R})^{\sim}, \delta^{\mathrm{sup}}(-) & \longleftrightarrow & \text{[IUTchIII], Theorem 3.11;} \\ SL_2(\mathbb{Z}), SL_2(\mathbb{Z})^{\sim}, \chi & \longleftrightarrow & \text{[IUTchIII], Corollary 3.12 } (\Longleftarrow \text{ Theorem 3.11}) \end{array}$$

— i.e., where, more precisely, the RHS of the latter correspondence is to be understood as referring to the derivation of [IUTchIII], Corollary 3.12, from [IUTchIII], Theorem 3.11. These last two correspondences are particularly interesting in light of the well-documented *historical fact* that the theory/estimates in Bogomolov's proof related to $SL_2(\mathbb{R})$, $SL_2(\mathbb{R})^{\sim}$, $\delta^{\mathrm{sup}}(-)$ were apparently *already known to Milnor* in the 1950's [cf. [MlWd]], while the idea of combining these estimates with the theory surrounding $SL_2(\mathbb{Z})$, $SL_2(\mathbb{Z})^{\sim}$, χ appears to have been unknown until the work of Bogomolov around the year 2000 [cf. [ABKP]].

Moreover, these last two correspondences — and, indeed, the entire analogy between Bogomolov's proof and inter-universal Teichmüller theory — are also of interest in the following sense:

> Bogomolov's proof only involves working with *elements* $\in SL_2(\mathbb{R}), SL_2(\mathbb{R})^{\sim}$ that arise from *topological fundamental groups*, hence may be applied *not only to* algebraic/holomorphic families of elliptic curves, but *also to arbitrary* **topological families** *of one-dimensional complex tori* that satisfy suitable conditions at the points of degeneration, i.e., "bad reduction".

This aspect of Bogomolov's proof is reminiscent of the fact that the *initial* Θ*-data* of inter-universal Teichmüller theory [cf. §3.3, (i)] essentially only involves data that arises from various **arithmetic fundamental groups** associated to an elliptic curve over a number field. In particular, this aspect of Bogomolov's proof suggests strongly that perhaps, in the future, some version of inter-universal Teichmüller theory could be developed in which the initial Θ-data of the current version of inter-universal Teichmüller theory is *replaced* by some *collection of topological groups* that satisfies conditions analogous to the conditions satisfied by the collection of arithmetic fundamental groups that appear in the initial Θ-data of the current version of inter-universal Teichmüller theory, but that does *not* necessarily arise, in a literal sense, from an elliptic curve over a number field.